

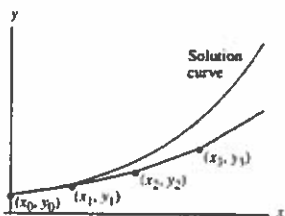
2.4: Numerical Approximation - Euler's Method

In Chapter 1, we saw a number of examples where the general first-order differential equation $y' = f(x, y)$ can be solved explicitly or exactly. However, this is the exception rather than the rule. For instance, the differential equation

$$\frac{dy}{dx} = e^{-x^2}$$

can only be solved with nonelementary functions and therefore we need a way of approximating the solution. Thus all the techniques from Chapter 1 are doomed to fail.

Example 1.



We wish to approximate the solution curve to $y' = d^{-x^2}$. We start with the initial point (x_0, y_0) and draw a small line segment of step size $\Delta x = h$ to get to the point (x_1, y_1) and repeat. You can see that our approximating is not great at first, but as $h \rightarrow 0$ we will get a good approximation of the solution curve. This is precisely the strategy of Euler's Method.

Euler's Method We start with the initial point (x_0, y_0) and solve for $y'(x_0, y_0) = f(x_0, y_0)$. We then repeat to get

$$\begin{array}{ll} x_1 = x_0 + h & y_1 = y_0 + h \cdot f(x_0, y_0) \\ x_2 = x_1 + h & y_2 = y_1 + h \cdot f(x_1, y_1) \\ x_3 = x_2 + h & y_3 = y_2 + h \cdot f(x_2, y_2) \\ x_4 = x_3 + h & y_4 = y_3 + h \cdot f(x_3, y_3) \\ \vdots & \vdots \end{array}$$

As an algorithm, with the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0,$$

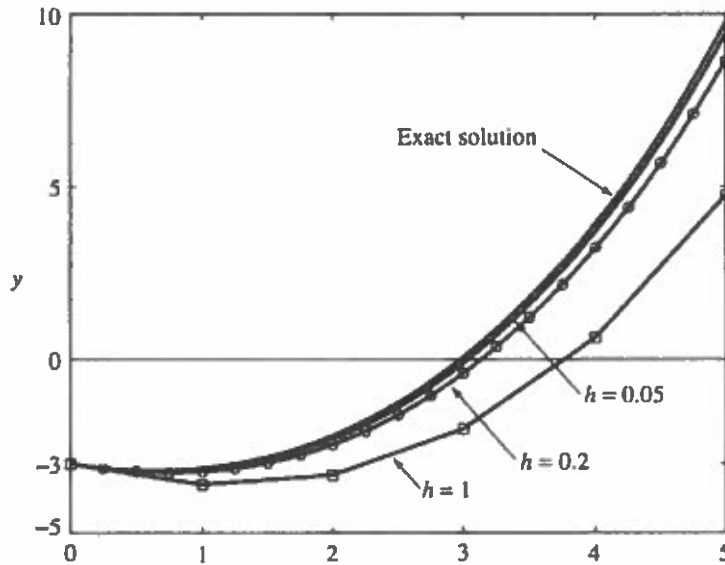
we get the algorithm for Euler's Method

$$x_{n+1} = x_n + h = x_0 + h \cdot (n + 1), \quad y_{n+1} = y_n + h \cdot f(x_n, y_n) \quad (n \geq 0).$$

Example 2. Use Euler's Method to approximate the solution of the initial value problem

$$\frac{dy}{dx} = x + \frac{1}{5}y, \quad y(0) = -3$$

first with step size $h = 1$ on the interval $[0,5]$ then with step size $h = 0.2$ on the interval $[0,1]$.



(a) $h=1$

$x_0 = 0$	$y_0 = -3$
$x_1 = 1$	$y_1 = -3.6$
$x_2 = 2$	$y_2 = -3.32$
$x_3 = 3$	$y_3 = -1.984$
$x_4 = 4$	$y_4 = 0.6192$
$x_5 = 5$	$y_5 \approx 4.7430$

~~Answer~~

(b) $h=0.2$

$x_0 = 0$	$y_0 = -3$
$x_1 = 0.2$	$y_1 = -3.12$
$x_2 = 0.4$	$y_2 \approx -3.205$
$x_3 = 0.6$	$y_3 \approx -3.253$
$x_4 = 0.8$	$y_4 \approx -3.263$
$x_5 = 1$	$y_5 \approx 3.234$